Random Lindblad equations from complex environments

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In this paper we demonstrate that Lindblad equations characterized by a random rate variable arise after tracing out a complex structured reservoir. Our results follows from a generalization of the Born-Markov approximation, which relies on the possibility of splitting the complex environment into a direct sum of subreservoirs, each one being able to induce by itself a Markovian system evolution. Strong non-Markovian effects, which microscopically originate from the entanglement with the different subreservoirs, characterize the average system decay dynamics. As an example, we study the anomalous irreversible behavior of a quantum tunneling system described in an effective two-level approximation. Stretched exponential and power law decay behaviors arise from the interplay between the dissipative and unitary hopping dynamics.

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I. INTRODUCTION

The dynamic of a small quantum system interacting weakly with uncontrollable degrees of freedom is well understood when a Markovian approximation applies. In this situation, after tracing out the environment, the system density matrix evolution can be well approximated by a Lindblad equation [1,2].

Although the applicability of the Markovian approximation ranges over many areas of physics [3–7] there exist several real systems whose dynamics present strong departures from it. The main signature of this departure is the presence of strong nonexponential decay behaviors, such as power law and stretched exponential. Some examples are nanocrystal quantum dots under laser radiation [8–10], superconducting qubits [11–13], spin environments [14], dephasing in atomic and molecular physics [15], electron transfer and exciton dynamics in proteins [16], and molecular systems maintained in a glassy environment [17], to name but a few. These and other specific experimental situations increase the necessity of finding formalism and effective evolutions able to describe the corresponding non-Markovian dynamics.

When the environment is modeled as an infinite set of normal modes, departure from a Markov approximation can be related to the corresponding spectral density function. This situation was extensively studied for the spin-boson and boson-boson models, where exact solutions are available [18–22]. Exact solutions can also be formulated for more general systems. Nevertheless, due to the huge analytical and numerical efforts needed for getting the non-Markovian system dynamics, alternative numerical methods based in a special decomposition of the spectral density function were formulated [23,24].

Anomalous system dynamics also arise from a random matrix modeling of the system-environment interaction [25–28]. This approach naturally applies when describing environments characterized by a complex dynamics such as for example chaotic ones.

Beyond the microscopic point of view, there exists an increasing interest in describing non-Markovian effects in open quantum systems by introducing memory contributions in Lindblad evolutions [29–32]. This procedure provides easily manageable dynamics. While most of these models are phenomenological, in this paper we will relate the presence of strong memory effects in the standard Lindblad theory [30] to the microscopic interaction of a system with a complex structured environment.

We will base our considerations on a splitting of the full Hilbert space of the bath into a direct sum of subreservoirs, constructed in such a way that each one guarantees the conditions for the applicability of a Markov approximation. Our motivation for formulating this splitting comes from systems embedded in glassy environments, where the underlying disorder produce localized bath states, inducing a natural shell structure of modes, each set having a different coupling strength with the system [17]. Thus, we associate a different Markovian sub-bath to each set of states.

As we will demonstrate, the splitting assumption allows us to generalize the usual Born-Markov approximation, which in a natural way leads to the formulation of Lindblad equations characterized by a random dissipative rate. As is well known from a classical context, master equations with random rates [33–36] are useful for describing strong non-Markovian effects [37,38]. Here we will demonstrate that the same scheme can also be applied in a quantum context under the previous conditions, i.e., a complex environment under the splitting condition.

As an example, we will study the anomalous dissipative dynamics of a quantum tunneling system described in a two-level approximation [18,21]. Strong nonexponential behaviors, such as stretched exponential and power law, arise from the interplay between the unitary hopping dynamics and the memory effects induced by the environment. The conditions under which our modeling can be mapped to the spin-boson model and stochastic dynamics are established. In this context, the differences between our framework and other approaches [23,24] introduced to deal with non-Markovian environments are established.

II. REDUCED DYNAMICS FROM COMPLEX ENVIRONMENTS

In general, the evolution of a system interacting with a complex environment cannot be described in a Markovian

approximation. While a general treatment is clearly not possible, when the splitting condition applies, for weak system-bath interactions, the reduced dynamics can be described through a generalization of the Born-Markov approximation.

A. Generalized Born-Markov approximation

We start by assuming a full microscopic Hamiltonian description

$$H_T = H_S + H_B + H_I, \tag{1}$$

where H_S correspond to the Hamiltonian of a system S, and H_B correspond to the Hamiltonian of the bath B. The term $H_I = q_S \otimes Q_B$ describes their mutual interaction, with the operators q_S and Q_B acting on the system and bath Hilbert spaces, respectively.

In an interaction representation with respect to $H_S + H_B$, the total density matrix $\rho_T(t)$ evolves as

$$\frac{d\rho_T(t)}{dt} = \frac{-i}{\hbar}[H_I(t), \rho_T(t)], \tag{2}$$

where $H_I(t)$ is the interaction Hamiltonian in the Heisenberg representation. Integrating formally this equation, and substituting the solution in it, the evolution of the reduced system density matrix $\rho_S(t) = \mathrm{Tr}_B\{\rho_T(t)\}$ can be written as

$$\frac{d\rho_S(t)}{dt} = -\left(\frac{1}{\hbar}\right)^2 \int_0^t dt' \mathrm{Tr}_B\{[H_I(t), [H_I(t'), \rho_T(t')]]\}, \tag{3}$$

where, as usual, a first order contribution was discarded after assuming $\mathrm{Tr}_B[H_I(t)\rho_T(0)] = 0$. From this evolution, the well known Born-Markov approximation can be deduced [3–6]. The Born approximation consists in assuming, at all times, an uncorrelated structure for the total density matrix

$$\rho_T(t) \approx \rho_S(t) \otimes \rho_B, \tag{4}$$

where ρ_B defines the stationary state of the bath. This assumption is consistent up to second order in the interaction Hamiltonian. When the decay of the bath correlation defines the small time scale of the problem, after introducing Eq. (4) in Eq. (3), the Markovian approximation leads to a closed density matrix evolution local in time.

We remark that the Born-Markov approximation does not rely in any specific model of bath dynamics [3–6], such as an infinite set of harmonic oscillators. In fact, its master equation is independent of model assumptions used in its derivation [7].

Now we consider a complex environment for which the previous approximations are not valid. As is usual when dealing with complex environments [25–28], instead of defining the bath Hamiltonian H_B as an infinite set of normal modes, here we specify it through its eigenstate basis $\{|\epsilon\rangle\}$, which in the weak interaction limit, is unmodified by the interaction with the system. As a central hypothesis, we will assume that, while the full action of the environment cannot be described in a Markov approximation, it is possible to split the full Hilbert space of the bath as a direct sum of subspaces, in such a way that each one defines a subreservoir able to induce by itself a Markovian system evolution [39]. These hypotheses are the main assumptions that allow us to formulate our results.

In conformity with the splitting condition, we write the interaction Hamiltonian as a direct sum of sub-Hamiltonians

$$H_I = H_{I_1} \oplus H_{I_2} \cdots \oplus H_{I_R} \oplus H_{I_{R+1}} \cdots, \tag{5}$$

where $H_{I_R} = q_S \otimes Q_{B_R}$. Here, each operator Q_{B_R} defines the interaction between the system and each subreservoir R.

In order to describe the joint action of all subreservoirs over the system, instead of the uncorrelated form Eq. (4), we introduce the *generalized Born approximation*

$$\rho_T(t) \approx \sum_R \rho_R(t) \otimes \Xi_R, \tag{6}$$

where $\mathrm{Tr}_S[\rho_R(t)] = 1$, and we have defined

$$\Xi_R = \sum_{\{\epsilon_R\}} \langle \epsilon_R|\rho_B|\epsilon_R\rangle |\epsilon_R\rangle\langle\epsilon_R|. \tag{7}$$

$\{|\epsilon_R\rangle\}$ is the base of eigenvectors that span the subspace corresponding to each subreservoir. Therefore, each contribution in Eq. (6) consists in an external product between a system state $\rho_R(t)$ and the projection of the stationary bath state ρ_B over each subspace R. In physical terms, each state $\rho_R(t)$ takes into account the dissipative effects induced by each subreservoir.

After introducing Eq. (6) in Eq. (3), we get the approximated evolution

$$\frac{d\rho_S(t)}{dt} \approx -\left(\frac{1}{\hbar}\right)^2 \sum_R P_R \int_0^t dt' \times \mathrm{Tr}_{B_R}\{[H_{I_R}(t), [H_{I_R}(t'), \rho_R(t') \otimes \rho_{B_R}]]\}, \tag{8}$$

where $\mathrm{Tr}_{B_R}\{\cdot\}$ means a trace operation with the states $\{|\epsilon_R\rangle\}$ corresponding to each subspace. Furthermore, we have introduced the sub-bath density matrix $\rho_{B_R} = \Xi_R / P_R$, where

$$P_R = \mathrm{Tr}_{B_R}\{\Xi_R\} = \sum_{\{\epsilon_R\}} \langle \epsilon_R|\rho_B|\epsilon_R\rangle. \tag{9}$$

The normalization condition $\mathrm{Tr}_B[\rho_B] = 1$ implies the relation $\Sigma_R P_R = 1$. Thus, the set $\{P_R\}$ can be seen as a set of probabilities defined by the weight of each subspace in the total stationary bath state.

From Eq. (6) we can write

$$\rho_S(t) = \mathrm{Tr}_B[\rho_T(t)] \approx \sum_R P_R \rho_R(t). \tag{10}$$

Then, the evolution Eq. (8) is in fact a linear combination of the evolutions corresponding to the set $\{\rho_R(t)\}$, each one participating with weight P_R. After introducing the *Markovian approximation* [3–6] to the evolution of each state $\rho_R(t)$, in a Schrödinger representation, we get

$$\frac{d\rho_R(t)}{dt} = \frac{-i}{\hbar}[H_S, \rho_R(t)] - \left(\frac{1}{\hbar}\right)^2 \int_0^\infty dt' \times \mathrm{Tr}_{B_R}\{[H_{I_R}, [H_{I_R}(-t'), \rho_R(t) \otimes \rho_{B_R}]]. \tag{11}$$

This evolution corresponds to the usual Born-Markov ap-

proximation when considering a bath consisting only of the subset of states $\{|\epsilon_R\rangle\}$ and characterized by the stationary state ρ_{B_R}. The system density matrix is defined by the linear combination Eq. (10).

B. Random Lindblad equations

The evolution Eq. (11), disregarding transients of the order of the sub-bath Hamiltonian correlation time, can be always well approximated by a Lindblad equation [1]

$$\frac{d\rho_R(t)}{dt} = \mathcal{L}_H[\rho_R(t)] + \gamma_R \mathcal{L}[\rho_R(t)], \tag{12}$$

where $\mathcal{L}_H[\cdot]=(-i/\hbar)[H_S,\cdot]$ is the system Liouville superoperator and the Lindblad superoperator is defined by

$$\mathcal{L}[\cdot] = \sum_\alpha \frac{1}{2}([V_\alpha, \cdot\, V_\alpha^\dagger] + [V_\alpha \cdot, V_\alpha^\dagger]). \tag{13}$$

As the underlying microscopic interaction between the system and the environment is the same in each subspace, the set of operators $\{V_\alpha\}$ does not depend on index R. Nevertheless, each subspace has associated a different characteristic dissipative rate γ_R. As this rate arises from the interaction of the system with the manifold of states $\{|\epsilon_R\rangle\}$, consistently with the Fermi golden rule [4], it is proportional to the characteristic interaction strength of each subspace, denoted as $|Q_{B_R}|$, multiplied by the corresponding density of states $g_R(\epsilon)=\Sigma_{\{\epsilon_R\}}\delta(\epsilon-\epsilon_R)$ evaluated at a characteristic frequency ω_S of the system, i.e., $\gamma_R \approx |Q_{B_R}|^2 g_R(\hbar\omega_S)$.

With these definitions in hand, we conclude that under the generalized Born-Markov approximation, we can represent the dynamics induced by the complex environment by a Lindblad master equation characterized by a random rate variable, defined by the set $\{\gamma_R, P_R\}$. Correspondingly, the system state follows from the average

$$\rho_S(t) = \sum_R P_R \rho_R(t) \equiv \langle \rho_R(t) \rangle. \tag{14}$$

Random rate equations were extensively used to model classical anomalous diffusion processes in disordered media [33–36]. Here, we have derived a similar structure for a different physical situation, i.e., quantum systems embedded in a complex structured environment.

As in a classical context, while the density matrices $\rho_R(t)$ follow a Markovian evolution, the system states $\rho_S(t)$ evolve with a non-Markovian evolution. This evolution can be easily obtained in a Laplace domain, where the average Eq. (14) takes the form

$$\rho_S(u) = \left\langle \frac{1}{u - (\mathcal{L}_H + \gamma_R \mathcal{L})} \right\rangle \rho_S(0) \equiv \langle G_R(u) \rangle \rho_S(0), \tag{15}$$

with u being the Laplace variable, and we have used the solutions $\rho_R(t)=\exp[(\mathcal{L}_H+\gamma_R\mathcal{L})t]\rho_S(0)$. Consistently with the uncorrelated initial condition $\rho_T(0)=\rho_S(0)\otimes\rho_B(0)$, there does not exist any statistical correlation between $\rho_S(0)$ and the random rate set. Thus, the average evolution can be obtained without appealing to a projector technique [33–36]. In fact, after introducing in Eq. (15) the identity in the form $\langle G_R(u)[u-(\mathcal{L}_H+\gamma_R\mathcal{L})]\rangle^{-1}$, it is immediate to get the deterministic, closed, non-Markovian evolution equation

$$\frac{d\rho_S(t)}{dt} = \mathcal{L}_H[\rho_S(t)] + \int_0^t d\tau\, \mathbb{L}(t-\tau)[\rho_S(\tau)], \tag{16}$$

where the superoperator $\mathbb{L}$ is defined in the Laplace domain by the equation

$$\langle G_R(u)\gamma_R\mathcal{L}\rangle[\cdot] = \langle G_R(u)\rangle \mathbb{L}(u)[\cdot]. \tag{17}$$

Depending on the set $\{\gamma_R, P_R\}$, Eq. (16) may lead to the presence of strong non-Markovian decay behaviors in the system dynamics. This characteristic originates from the entanglement of the system with each subreservoir, situation explicitly introduced by Eq. (6).

An example of a complex structured environment where the generalized Born-Markov applies straightforwardly is a bath Hamiltonian whose eigenvectors can be labeled with two indexes (E,R). The index E is continuous, and for each R the corresponding submanifold of states is able to induce a different system Markovian decay. The difference between the Markovian dynamics may originate in the coupling strength of each submanifold with the system. On the other hand, it may originate due to strong variations of the bath density of states with index R. The system dynamics follows as a superposition of Markovian dynamics whose weights are taken into account through the generalized Born approximation Eq. (6). If the decay induced by each submanifold is the same, the generalized Born approximation reduces to the usual one, and then a Markovian evolution is obtained. Further examples can be established in the context of random band-matrix models [7], where the Markovian sub-baths, for example, may be associated with subspaces with a different characteristic bandwidth.

C. Effective approximation

Classical master equations with random rates are characterized by equations similar to those obtained previously. Nevertheless, as in general the underlying numbers of states is infinite, some kind of approximation is necessary in order to obtain the operator $\mathbb{L}$, as for example an effective medium approximation [35,36]. Here, we introduce a similar approximation in order get a general characterization of the dynamics. Thus, in Eq. (17) we discard the dependence introduced by the Lindblad superoperator $\mathcal{L}$ in the propagator $G_R(u)$, i.e., $\mathcal{L}\to -\mathrm{I}$. Then, we get the approximated solution $\mathbb{L}(u) \simeq K(u-\mathcal{L}_H)\mathcal{L}$, from where follows the evolution

$$\frac{d\rho_S(t)}{dt} \simeq \mathcal{L}_H[\rho_S(t)] + \int_0^t d\tau\, K(t-\tau) e^{(t-\tau)\mathcal{L}_H} \mathcal{L}[\rho_S(\tau)]. \tag{18}$$

In this approximation all information about the random rate is introduced through the kernel function

$$K(u) = \left\langle \frac{\gamma_R}{u+\gamma_R} \right\rangle \left\langle \frac{1}{u+\gamma_R} \right\rangle^{-1}. \quad (19)$$

As in a classical context, this kernel can be associated with a waiting time distribution $w(t)$ and a survival probability $P_0(t)$ defined by

$$w(u) = \left\langle \frac{\gamma_R}{u+\gamma_R} \right\rangle, \quad P_0(u) = \left\langle \frac{1}{u+\gamma_R} \right\rangle. \quad (20)$$

In classical master equations, these objects define a continuous time random walk [33–38]. In the quantum case, a similar stochastic dynamics can be constructed [30]. It consists in the application at random times of the superoperator $\mathcal{E}=\mathcal{L}+\mathrm{I}$, implying the transformation $\rho \to \mathcal{E}[\rho]$, while during the intervals between these disruptive actions the system evolves with its unitary dynamics $U(t)=\exp[t\mathcal{L}_H]$. The intervals between the successive applications of $\mathcal{E}$ follow from the waiting time distribution $w(t)$. The function $P_0(t)$ defines the corresponding survival probability $P_0(t)=1-\int_0^t d\tau\, w(\tau)$. Thus, the average over different realizations of the random times can be written as

$$\rho_S(t) = P_0(t)e^{t\mathcal{L}_H}\rho_S(0) + \int_0^t d\tau\, w(t-\tau)e^{(t-\tau)\mathcal{L}_H}\mathcal{E}[\rho_S(\tau)]. \quad (21)$$

From here, in the Laplace domain, it is straightforward to recuperate the evolution Eq. (18). When $\mathcal{L} \neq \mathcal{E}-\mathrm{I}$, with $\mathcal{E}$ a completely positive superoperator [2], a similar stochastic dynamics can be formulated after introducing a limit procedure [40].

We remark that the stochastic interpretation [Eq. (21)] was constructed after associating with the kernel $K(t)$ a waiting time distribution and a survival probability Eq. (20). This association does not rely on the generalized Born-Markov approximation, neither was it deduced from a conditional continuous time measurement theory [5]. Therefore, it is not clear if one can associate with the stochastic dynamics a random signal of a measurement apparatus. If this is the case, contradictions between environmental decoherence and wave-function collapse may arise [41,42].

III. QUANTUM TUNNELING SYSTEM DRIVEN BY A COMPLEX ENVIRONMENT

As an example of our formalism, in this section we will characterize the dissipative dynamics of a quantum tunneling system described in a two-level approximation [18,21] and driven by a complex environment. Then, the system Hamiltonian can be written as

$$H_S = \frac{\hbar\omega_A}{2}\sigma_z + \frac{\hbar\Delta}{2}\sigma_x. \quad (22)$$

The first term, proportional to the z Pauli matrix σ_z, defines the energy of the effective levels, and the second one, proportional to the x Pauli matrix σ_x, introduce the reversible hopping between the two effective states.

The complex environment will be represented by the Lindblad superoperator

$$\mathcal{L}[\cdot] = \frac{1}{2}([\sigma_z \cdot, \sigma_z] + [\sigma_z, \cdot\, \sigma_z]), \quad (23)$$

and an arbitrary set $\{\gamma_R, P_R\}$ of random rates and weights. For fixed rate, this superoperator induces a dynamics equivalent to a thermal environment in the high temperature limit [22].

The evolution of the system density matrix is defined by Eqs. (16) and (17). Here, we write the evolution in terms of the components of the Bloch vector, which are defined by the mean value of the Pauli matrices, $S_j(t)=\mathrm{Tr}_S\{\rho_S(t)\sigma_j\}$, with $j=x,y$, and z. We get

$$\frac{dS_X(t)}{dt} = -\omega_A S_Y(t) - \int_0^t d\tau\{\Gamma_X(t-\tau)S_X(\tau) - \Upsilon(t-\tau)S_Y(\tau)\}, \quad (24a)$$

$$\frac{dS_Y(t)}{dt} = \omega_A S_X(t) - \Delta S_Z(t) - \int_0^t d\tau \times \{\Gamma_Y(t-\tau)S_Y(\tau) + \Upsilon(t-\tau)S_X(\tau)\}, \quad (24b)$$

$$\frac{dS_Z(t)}{dt} = \Delta S_Y(t). \quad (24c)$$

Thus, the system evolution is completely characterized by three memory kernels $\Gamma_X(t), \Gamma_Y(t)$, and $\Upsilon(t)$. In Appendix A, we give the exact expressions of these kernels for arbitrary random rates, along with the kernels that arise from the effective approximation Eq. (18). From Eq. (24a)–(24c) it is straightforward to write the system density matrix evolution [Eq. (16)] as a sum of Lindblad superoperators, each one characterized by a different kernel.

A. Dispersive limit

When the hopping frequency is zero, $\Delta=0$, the dynamics reduces to a dispersive one. Thus, the coherences decay continuously while the population of each effective level remains constant. In this limit, from Appendix A, for an arbitrary set $\{\gamma_R, P_R\}$ we get the exact kernels

$$\Gamma_X(t) = K(t)\cos[\omega_A t], \quad (25a)$$

$$\Gamma_Y(t) = K(t)\cos[\omega_A t], \quad (25b)$$

$$\Upsilon(t) = K(t)\sin[\omega_A t], \quad (25c)$$

where $K(t)$ is defined in the Laplace domain by Eq. (19). We note that these kernels also arise from the effective approximation Eq. (18), indicating that for $\Delta=0$, both evolutions coincide.

From Eqs. (24a)–(24c), the exact solution of the Bloch vector is given by

$$S_X(t) = P_0(t)\{\cos[\omega_A t]S_X(0) - \sin[\omega_A t]S_Y(0)\}, \quad (26a)$$

$$S_Y(t) = P_0(t)\{\sin[\omega_A t]S_X(0) + \cos[\omega_A t]S_Y(0)\}, \quad (26b)$$

$$S_Z(t) = S_Z(0), \tag{26c}$$

where $P_0(t)$ is the survival probability defined by its Laplace transform Eq. (20), which in the time domain reads $P_0(t) = \Sigma_R P_R \exp[-\gamma_R t]$. Consistently, we note that the exact solutions Eqs. (26a)–(26c) correspond to an average over Markovian solutions, each one characterized by a rate γ_R and participating with weight P_R. Depending on the distribution of the dissipation rate, arbitrary forms of the decay can be obtained from this average over exponential functions. Hence the non-Markovian behavior can be observed in the relaxation of the density matrix to the stationary state.

B. Anomalous decay behaviors

The form of the set $\{\gamma_R, P_R\}$ depends on the specific structure of the complex environment. Here, we will determine this set in a phenomenological way as a function of the system decay behavior. We will be interested in obtaining anomalous decay dynamics such as a power law. A possible set consistent with this decay is

$$\gamma_R = \gamma_0 \exp[-bR], \quad P_R = (1 - e^{-a}) \exp[-aR], \tag{27}$$

where $R \in [0, \infty]$, γ_0 scale the random rates, and the constants b and a measure the exponential decay of the random rates and their corresponding weights. With these definitions, it is simple to demonstrate that after a transient of order γ_0, the waiting time distribution and its associated survival probability Eq. (20) present a power law decay behavior [43], $w(t) \approx 1/(\gamma_0 t)^{1+\alpha}$, and $P_0(t) \approx 1/(\gamma_0 t)^{\alpha}$, where $\alpha = a/b$. Clearly, this behavior is reflected in the system dynamics.

When $0 < \alpha < 1$, the kernel $K(t)$ corresponding to the set Eq. (27) can be well approximated by the expression

$$K(u) \simeq \frac{\gamma}{1 + (\beta/u)^{1-\alpha}}, \tag{28}$$

with the definitions

$$\gamma = \langle \gamma_R \rangle, \quad \beta = \frac{\langle \gamma_R^2 \rangle - \langle \gamma_R \rangle^2}{\langle \gamma_R \rangle}. \tag{29}$$

The scaling of these parameters can be motivated by considering a two-dimensional set of random rates [44]. From Eqs. (19) and (20), the waiting time distribution and its associated survival probability can be obtained as

$$w(u) = \frac{K(u)}{u + K(u)}, \quad P_0(u) = \frac{1}{u + K(u)}. \tag{30}$$

From here, is it is simple to prove that $w(u)$ is a completely monotonic function [30], which implies that $P_0(t)$ decays in a monotonic way or equivalently, $w(t) \geqslant 0$.

In Fig. 1 we plot the survival probability $P_0(t)$ by assuming the kernel Eq. (28) for different values of β/γ. We note that in the short time regime, the decay is an exponential one, while in the asymptotic regime a power law behavior is present,

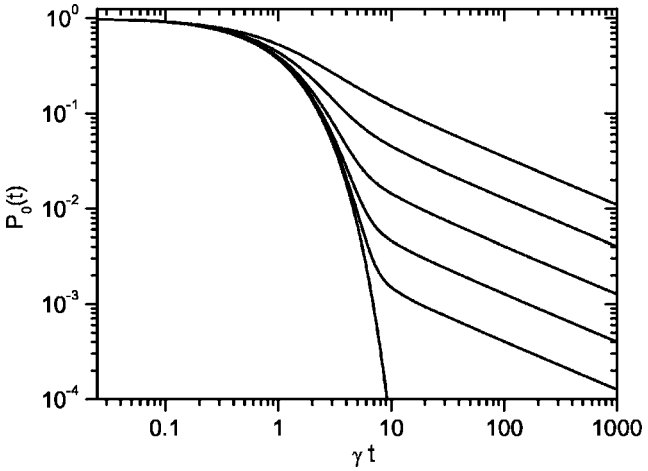

FIG. 1. Survival probability. From top to bottom, the parameters are $\beta/\gamma = 0.75,\ 10^{-1}, 10^{-2}, 10^{-3}, 10^{-4}$, and the Markovian limit $\beta = 0$. In all cases we take $\alpha = 1/2$.

$$P_0(t) \simeq \exp[-\gamma t], \quad P_0(t) \simeq \frac{\beta^{1-\alpha}}{\gamma \Gamma(1-\alpha)} \frac{1}{t^{\alpha}}, \tag{31}$$

where $\Gamma(x)$ is the gamma function. These asymptotic behaviors follow immediately from Eq. (30). When the dispersion of the random rate γ_R is zero ($\beta = 0$), consistently the dynamics reduces to a Markovian one, $K(u) = \gamma$, which implies the pure exponential decay $P_0(t) = \exp[-\gamma t]$ and $w(t) = \gamma \exp[-\gamma t]$.

In the next section we will characterize the tunneling dynamics by assuming a complex environment characterized by the random rate set Eq. (27) or equivalently by the kernel Eq. (28).

C. Tunneling dynamics

Here we will analyze the tunneling dynamics for the symmetric case $\omega_A = 0$, which arises when the two effective levels have the same energy. From Appendix A, the exact kernels read

$$\Gamma_X(u) = K(u), \tag{32a}$$

$$\Gamma_Y(u) = K(u + \Delta^2/u), \tag{32b}$$

$$\Upsilon(u) = 0. \tag{32c}$$

As before, the kernel $K(u)$ is defined by Eq. (19). The exact solution of the Bloch vector can be obtained in the Laplace domain. We get

$$S_X(u) = \frac{1}{u + \Gamma_X(u)} S_X(0), \tag{33a}$$

$$S_Y(u) = \Lambda(u)\{u S_Y(0) - \Delta S_Z(0)\}, \tag{33b}$$

$$S_Z(u) = \Lambda(u)\{[u + \Gamma_Y(u)] S_Z(0) + \Delta S_Y(0)\}, \tag{33c}$$

where we have defined

$$\Lambda(u) = \frac{1}{u^2 + u\Gamma_Y(u) + \Delta^2}, \tag{34}$$

which can also be expressed as $\Lambda(u) = P_0(u + \Delta^2/u)/u$.

In this case it is not possible to find in the time domain a general exact solution for arbitrary memory kernels. A simple analytical solution is only available in the Markovian case $[K(u) = \gamma]$,

$$S_X(t) = e^{-\gamma t} S_X(0), \tag{35a}$$

$$S_Y(t) = e^{-\gamma t/2}\{S_Y(0)\cosh[\lambda t] - \lambda^{-1}[(\gamma/2)S_Y(0) + \Delta S_Z(0)]\sinh[\lambda t]\}, \tag{35b}$$

$$S_Z(t) = e^{-\gamma t/2}\{S_Z(0)\cosh[\lambda t] + \lambda^{-1}[(\gamma/2)S_Z(0) + \Delta S_Y(0)]\sinh[\lambda t]\}, \tag{35c}$$

where $\lambda = \sqrt{(\gamma/2)^2 - \Delta^2}$, and γ defines the unique dissipative rate. Notice that in the limit of null dissipation, a periodic hopping between the effective levels is obtained.

For arbitrary random rates $\{\gamma_R, P_R\}$, the dynamics can be characterized in different regimes. First, in the case $\Delta \gg \langle \gamma_R \rangle$, from Eqs. (33a)–(33c) it is possible to get the approximated solutions

$$S_X(t) = P_0(t) S_X(0), \tag{36a}$$

$$S_Y(t) \simeq P_0(t/2)\{\cos[\Delta t] S_Y(0) - \sin[\Delta t] S_Z(0)\}, \tag{36b}$$

$$S_Z(t) \simeq P_0(t/2)\{\sin[\Delta t] S_Y(0) + \cos[\Delta t] S_Z(0)\}. \tag{36c}$$

Thus, the dynamics consists in a periodic tunneling between the two effective states, and whose decay can be written in terms of the survival probability. As in the previous case, this solution corresponds to an average over the corresponding Markovian solutions, i.e., Eq. (35a)–(35c) written in the limit of small decay rate when compared to the tunneling frequency Δ.

In Fig. 2 we plot the average of the z Pauli matrix which follows from Eq. (33a)–(33c) with the kernel Eq. (28). As initial condition we take the upper eigenstate of σ_z. We verified that the exact solutions are well described by the approximation Eq. (36a)–(36c) for parameter values satisfying $\gamma/\Delta \lesssim 1$. As the envelope decay is given by $P_0(t/2)$, by increasing the average rate γ, the dynamics decays in a faster way. This dependence is broken when the average rate is much greater than the hopping frequency.

In the limit $\Delta \ll \langle \gamma_R \rangle$, the dissipative dynamics dominates over the tunneling one. In Fig. 3 we plot $S_Z(t)$ [Eq. (33a)–(33c)] for different values of the characteristic parameters of the kernel Eq. (28). We note that by increasing the average rate γ, a slower decay is obtained. Thus, the dynamics develops a Zeno-like effect [45,46]. From the exact solution Eq. (33a)–(33c), the characteristic decay of the Bloch vector can be approximated by the expressions

$$S_X(u) = P_0(u) S_X(0), \tag{37a}$$

$$S_Y(u) \simeq \mathcal{Z}(u)\{u S_Y(0) - \Delta S_Z(0)\} d(u), \tag{37b}$$

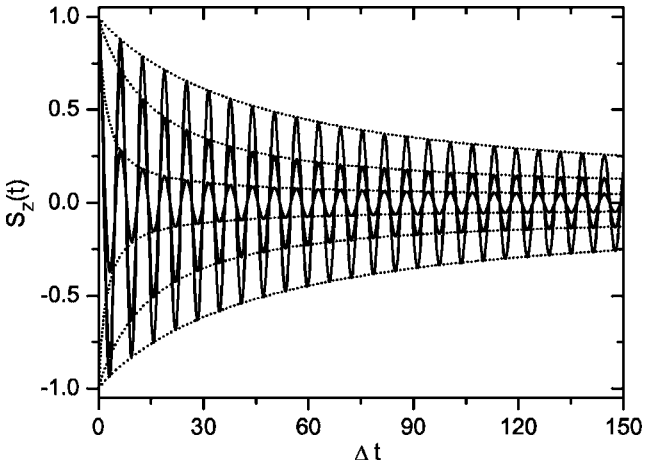

FIG. 2. Average $S_Z(t)$ considering as initial condition the upper eigenstate of σ_z. The envelopes are given by $\pm P_0(t/2)$. From top to bottom, the parameters are $\gamma/\Delta = 0.05$, 0.15, and 1.0. In all cases we take $\alpha = 1/2$, $\beta = \gamma/2$, and $\omega_A = 0$.

$$S_Z(u) \simeq \mathcal{Z}(u)\{S_Z(0) + \Delta S_Y(0) d(u)\}, \tag{37c}$$

where we have introduced $\mathcal{Z}(u) = u^{-1} w(\Delta^2/u)$ and the function $d(u) = [u + K(\Delta^2/u)]^{-1} \simeq u^{-1} P_0(\Delta^2/u)$. For the kernel defined by Eq. (28), the characteristic decay $\mathcal{Z}(u)$ is given by

$$\mathcal{Z}(u) = \frac{1}{u + C_1 + C_\alpha u^{1-\alpha}}, \quad C_\alpha = \frac{\beta^{1-\alpha}\Delta^{2\alpha}}{\gamma}. \tag{38}$$

As can be seen in Fig. 3 (dotted line), disregarding the oscillatory behavior, after the transient $\gamma t \ll 1$, this function provides an excellent fitting of the decay dynamics.

The function $\mathcal{Z}(t)$ is characterized by a reach variety of behaviors. First, we note that in the Markovian limit, $\beta = 0$, we get an exponential decay with rate $C_1 = \Delta^2/\gamma$, which clearly diminishes by increasing γ. In the non-Markovian case, in the short time regime, we can approximate

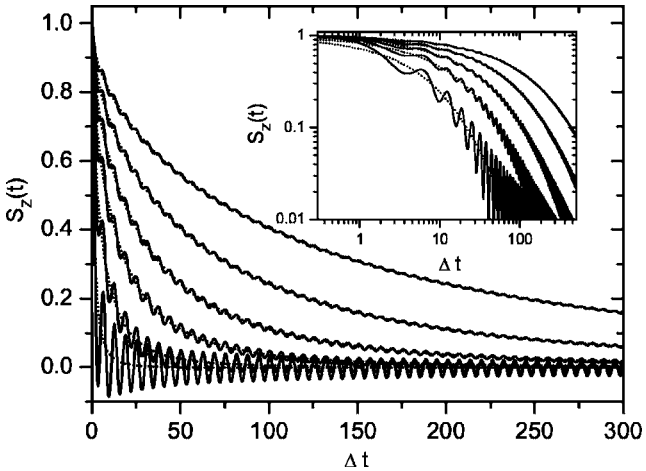

FIG. 3. Average $S_Z(t)$ considering as initial condition the upper eigenstate of σ_z. The fitting decay curves (dotted lines) are given by Eq. (38). From top to bottom, the parameters are $\gamma/\Delta = 200$, 100, 50, 25, 10, and 2.5. In all cases we take $\alpha = 1/2$, $\beta = \gamma/2$, and $\omega_A = 0$. In the inset we show the same graphic in a log-log scale.

$$\mathcal{Z}(t) \simeq \exp\left[-\left(C_1 t + \frac{C_\alpha t^\alpha}{\Gamma(1+\alpha)}\right)\right], \tag{39}$$

while in the asymptotic long time limit we get

$$\mathcal{Z}(t) \simeq \frac{(1-\alpha)}{\Gamma(\alpha)} \frac{C_\alpha}{C_1^2} \frac{1}{t^{2-\alpha}}. \tag{40}$$

Thus, the dispersion of the random rate (measured by β) induces, at short times, an extra stretched exponential decay, while in the asymptotic regime it scales the power law behavior $[C_1^2/C_\alpha = \Delta^{2(2-\alpha)}/\gamma\beta^{1-\alpha}]$. The characteristic rates of both regimes arise from a competence between the unitary and dissipative dynamics. We notice that by increasing the dispersion rate β, the characteristic rate of the stretched exponential decay is increased, while the rate for the power law regime is decreased. The dependence on the hopping frequency Δ is the inverse one.

The Zeno-like effect can be *qualitatively* understood in terms of the stochastic evolution corresponding to the effective approximation Eq. (18). This stochastic process develops in the system Hilbert space and consists in the application at random times of the superoperator $\mathcal{E} = \mathcal{L} + \mathrm{I}$, which in view of Eq. (23) reads $\mathcal{E}[\cdot] = \sigma_z \cdot \sigma_z$, while in the intermediates times the system evolves with its unitary evolution $U(t) = \exp[-i\Delta t \sigma_x/2]$. The superoperator $\mathcal{E}$ implies the disruptive transformations $S_X \to -S_X$, $S_Y \to -S_Y$, $S_Z \to S_Z$, while the unitary dynamics is equivalent to a rotation around the x direction. In the limit of vanishing hopping frequency Δ, the continuous applications of the superoperator $\mathcal{E}$ destroy the x-y components and freeze the dynamics in the initial condition $S_Z(0)$. Thus, a pure Zeno effect is recovered. For $\Delta/\langle\gamma_R\rangle \ll 1$, the decay dynamics is determined from the competition between the transformations induced by $\mathcal{E}$ and $U(t)$, defining the Zeno-like regime. This interpretation is exact in the Markovian limit and always valid for the effective master equation Eq. (18).

D. Anomalous decay behavior from a finite set of random rates

In obtaining the previous results we have assumed an infinite set of random rates Eq. (27), whose effects can be approximated by the kernel Eq. (28). While this election guarantees the presence of an asymptotic power law decay, strong nonexponential behaviors can be obtained in an *intermediate regime* by considering only a finite set, $1 < R \leqslant N_{\max}$, of random rates $\{\gamma_R, P_R\}$. On the other hand, for a finite set, the asymptotic system dynamics is always Markovian and characterized by the inverse rate $\langle 1/\gamma_R\rangle$. This result follows from $\lim_{u\to 0} K(u) = \langle 1/\gamma_R\rangle$.

In Fig. 4 we show the decay dynamics induced by an environment characterized by a finite set of random rates γ_R ($N_{\max}=7$) with equal weights, $P_R = 1/N_{\max}$. Each curve follows from a superposition of Markovian solutions, Eq. (35a)–(35c) with $\gamma \to \gamma_R$. The set of rates $\{\gamma_R\}$ of each plot differ in a multiplicative factor, in such a way that the relation between the average rate $\gamma = \langle\gamma_R\rangle$ and the corresponding fluctuation rate $\beta = [\langle\gamma_R^2\rangle - \langle\gamma_R\rangle^2]/\langle\gamma_R\rangle$ remains constant in all

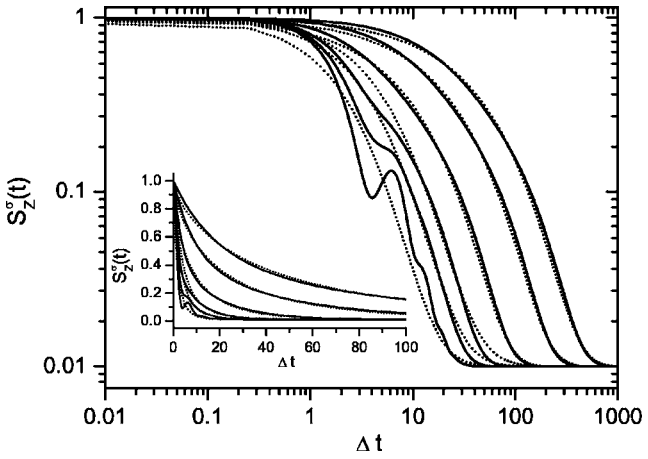

FIG. 4. Average $S_Z^\sigma(t) = \sigma + (1-\sigma)S_Z(t)$, considering as initial condition the upper eigenstate of σ_z and a finite set of random rates with equal weights. From top to bottom, the parameters are $\gamma/\Delta = 50$, 25, 10, 5, 3.5, and 2.5. In all cases we take $\sigma = 0.01$, $\beta/\gamma = 0.51$, and $\omega_A = 0$. For $\gamma/\Delta \geqslant 10$, the dotted lines correspond to the fitting $S_Z^\sigma(t) \simeq \sigma + (1-\sigma)\exp[-(\zeta t)^\delta]$, while for $\gamma/\Delta \leqslant 10$, they correspond to $S_Z^\sigma(t) \simeq \sigma + (1-\sigma)[1+\zeta t]^{-\delta}$ (see text).

curves. For the case $\gamma/\Delta = 2.5$, the random rates are $\gamma_R/\Delta = 0.59$, 1.0, 1.09, 1.21, 4.0, 4.7, and 4.88.

In order to shed light on the intermediate nonexponential regime, we have plotted the shifted average $S_Z^\sigma(t) = \sigma + (1-\sigma)S_Z(t)$, with $\sigma \ll 1$. In the deep Zeno-like regime ($\gamma/\Delta \gtrsim 10$), $S_Z^\sigma(t)$ can be well approximated by a stretched exponential behavior $S_Z^\sigma(t) \simeq \sigma + (1-\sigma)\exp[-(\zeta t)^\delta]$, with $\delta \approx 0.7$ and $\zeta/\Delta \in (0.025, 0.12)$. For $\gamma/\Delta \lesssim 10$, a power law fitting is more adequate $S_Z^\sigma(t) \simeq \sigma + (1-\sigma)[1+\zeta t]^{-\delta}$, with $\delta \in (3.5, 4.5)$ and $\zeta/\Delta \in (0.05, 0.125)$. We note that a similar nonexponential fitting was found in Ref. [17] by considering the action of a finite bath, which can be associated with a glassy environment. On the other hand, the oscillatory effects in the decay of $S_Z^\sigma(t)$ arise from the Markovian solutions Eq. (35a)–(35c), corresponding to the rates satisfying $\gamma_R/\Delta \leqslant 2$. In fact, for the Markovian solution, this condition delimits the change between a monotonic and an oscillatory decay behavior. Consistently, we notice that by increasing the average rate, the amplitude of the oscillations are smaller. A similar effect can be seen in Fig. 3.

E. Mapping with other models

Our formalism relies on the applicability of the generalized Born-Markov approximation. Here we explore the possibility of mapping its dynamics with other models that also induce anomalous decay behaviors.

Spin-boson model. The spin-boson model is defined by the total Hamiltonian

$$H_T = \frac{\hbar}{2}\{\omega_A \sigma_z + \Delta\sigma_x\} + \frac{\hbar}{2}\sigma_z Q_B + H_B, \tag{41}$$

where the bath Hamiltonian $H_B = \Sigma_j[(p_j^2/2m_j) + m_j\omega_j^2 q_j^2]$ corresponds to a set of harmonic oscillators, and $Q_B = d\Sigma_j \kappa_j q_j$. The bath is characterized by the spectral density function

$$J(w) = (\pi/2) d \sum_j (\kappa_j^2/m_j w_j)\, \delta(w - w_j), \tag{42}$$

and assumed to be in equilibrium at temperature T. As is well known, the reduced system dynamics can be obtained in an exact way [18–22]. It reads

$$S_X(t) = \int_0^t d\tau [Y_A^{(s)}(t-\tau) + Y_A^{(a)}(t-\tau) S_Z(\tau)] + Y_B^{(s)}(t) S_X(0) + Y_B^{(a)}(t) S_Y(0), \tag{43a}$$

$$S_Y(t) = \frac{1}{\Delta} \frac{dS_Z(t)}{dt}, \tag{43b}$$

$$\frac{dS_Z(t)}{dt} = \int_0^t d\tau [K_A^{(a)}(t-\tau) - K_A^{(s)}(t-\tau) S_Z(\tau)] + K_B^{(a)}(t) S_X(0) + K_B^{(s)}(t) S_Y(0), \tag{43c}$$

where the corresponding kernels can be written as functions of $J(w)$. On the other hand, it is possible to write the exact averaged evolution Eq. (24a)–(24c) in the form Eqs. (43a)–(43c). In Appendix B we present the kernels corresponding to each dynamics. From these expressions, it is simple to prove that to first order in Δ, after disregarding a phase factor, both sets of kernels can be mapped under the condition

$$\sum_R P_R \exp[-\gamma_R t] = \exp[-Q'(t)], \tag{44}$$

where

$$Q'(t) = \frac{d^2}{\hbar \pi} \int_0^\infty dw \frac{J(w)}{w^2} \coth\left(\frac{\hbar w}{2kT}\right) [1 - \cos(wt)] \tag{45}$$

defines the real part of the thermal bath correlation. We remark that the mapping Eq. (44) is valid only in the high temperature limit, a condition consistent with the Lindblad structure Eq. (23).

In this context, from Eq. (44), it is possible to shed light on the difference between the present approach and that developed in Refs. [23,24]. In our approach, which relies on splitting the Hilbert space of the bath as a *direct sum* of subspaces, $\exp[-Q'(t)]$ is written as a sum of exponential functions, each one associated with each Markovian subreservoir. Instead, in Refs. [23,24], $Q'(t)$ is expressed as a sum of exponential functions. This representation relies on an artificial decomposition of the spectral density function $J(w)$ as a sum of individual terms. Thus, the Hilbert space of the bath is effectively split into an *external product* of subspaces, each one associated with a non-Markovian subreservoir. As in our approach, the system density matrix can be written in terms of a set of auxiliary subdensity matrixes. Nevertheless, their evolution involves coupling among them all.

Stochastic Hamiltonian. Decoherence in small quantum systems is also modeled by introducing stochastic elements in the system evolution. This situation arises naturally in many physical systems [11–13,47]. Consistently with the spin-boson model we consider the stochastic Hamiltonian

$$H_{st}(t) = \frac{\hbar}{2} \{[\omega_A + \xi(t)] \sigma_z + \Delta \sigma_x\}, \tag{46}$$

where $\xi(t)$ is a classical nonwhite noise term.

By assuming $\langle\langle \xi(t) \rangle\rangle_\xi = 0$, where $\langle\langle \cdots \rangle\rangle_\xi$ means an average over realizations of the noise, in the limit of vanishing Δ it is simple to solve the stochastic dynamics and obtain the average of the Pauli matrixes. The final evolution is the same as in Eq. (26a)–(26c) after replacing $P_0(t)$ with the average dephasing factor $D(t) = \langle\langle \exp[i \int_0^t d\tau\, \xi(\tau)] \rangle\rangle_\xi$. Thus, the generalized Born-Markov approximation can be mapped to the stochastic Hamiltonian evolution under the condition $D(t) = P_0(t)$, which explicitly reads

$$\sum_R P_R \exp[-\gamma_R t] = \left\langle\left\langle \exp\left[i \int_0^t d\tau\, \xi(\tau) \right] \right\rangle\right\rangle_\xi . \tag{47}$$

This condition can be consistently satisfied if the dephasing factor $D(t)$ decays in a monotonic way.

IV. SUMMARY AND CONCLUSIONS

We have presented a theoretical approach intended to describe the dynamics of small quantum systems interacting with a complex structured environment. Our formalism is based in an extension of the well known Born-Markov approximation, which relies on the possibility of splitting the environment as a direct sum of subreservoirs, each one being able to induce by itself a Markovian system dynamics. Then we have demonstrated that the full action of the complex environment can be described through a random Lindblad master equation. The set of random rates follows from a Fermi golden rule. Thus, they are proportional to the characteristic coupling strength of each subspace multiplied by the corresponding subdensity of states evaluated in a characteristic frequency of the system. The associated probabilities are defined by the weight of each subspace in the stationary state of the bath.

From a phenomenological point of view, the set of random rates and weights can be determined in a consistent way in function of the system decay. In fact, the system dynamics is characterized by a non-Markovian master equation that as a function of the random rate set can develop strong nonexponential decays.

As an example we worked out the dissipative dynamics of a quantum tunneling system in the two-level approximation. We have introduced a set of random rates that lead to the presence of asymptotic power law decay. In the limit of small hopping frequency, when compared with the average rate, we have showed that a Zeno-like phenomenon arises, which is characterized by a stretched exponential and a power law decay. These behaviors follow from the interplay between the unitary dynamics and the entanglement-memory effects induced by the reservoir.

For the tunneling dynamics, we have also demonstrated that nonexponential decays arise even by considering a small set of random rates. Furthermore, we have established the conditions under which the random Lindblad evolution can be mapped to a spin-boson model and a stochastic Hamiltonian evolution.

Finally, we want to emphasize that the present results define a framework for describing anomalous quantum system dynamics, which consists in taking the characteristic rate of a Lindblad equation as a random distributed variable. We remark that this approach was not derived from an ensemble of identical systems whose local interactions with the environment can be approximated by different Markovian evolutions. In fact, the underlying microscopic physics can be related to a *single* quantum system coupled to an environment with a complex structured spectral density function and whose dynamical influence over the system can be approximated by a direct sum of Markovian subreservoirs. Thus, our approach may be relevant for the description of anomalous decay processes in individual mesoscopic systems embedded in a condensed phase environment [8–10]. A natural example for which the generalized Born-Markov approximation may apply are glassy reservoirs, where the underlying configurational disorder produces a hierarchical distribution of coupling strength between the single system and the corresponding localized eigenstates of the reservoir [17].

APPENDIX A: EXACT KERNELS

Here we present the exact expressions for the kernels $\Gamma_X(u), \Gamma_Y(u)$, and $\Upsilon(u)$ that define the evolution of the Pauli operators average Eqs. (24a)–(24c). For arbitrary rates $\{\gamma_R, P_R\}$, the kernels read

$$\Gamma_X(u) = D\{[u(u+C)+\Delta^2](u+B)+u\omega_A^2\}, \quad \text{(A1a)}$$

$$\Gamma_Y(u) = D\{[u(u+B)+\Delta^2](u+C)+u\omega_A^2\}, \quad \text{(A1b)}$$

$$\Upsilon(u) = D(B-C)u\omega_A, \quad \text{(A1c)}$$

where D denotes the function

$$D(u) = \frac{B(u)}{\{u[u+B(u)]+\Delta^2\}[u+B(u)]+u\omega_A^2}. \quad \text{(A2)}$$

The extra functions B and C are defined by

$$B(u) = \frac{\langle G(u)\gamma_R\rangle}{\langle G(u)\rangle}, \quad C(u) = \frac{\langle G(u)\gamma_R^2\rangle}{\langle G(u)\gamma_R\rangle}, \quad \text{(A3)}$$

where we have introduced

$$G(u) = \frac{1}{[u(u+\gamma_R)+\Delta^2](u+\gamma_R)+u\omega_A^2}. \quad \text{(A4)}$$

Using that the Laplace transform of $f(t)e^{\pm i\omega_A t}$ is given by $f(u\mp i\omega_A)$, in the case $\Delta=0$ it is possible to regain the expressions of Sec. III A, Eqs. (25a)–(25c). On the other hand, taking $\omega_A=0$ it is straightforward to get the results of Sec. III C, Eqs. (32a)–(32c).

In an effective approximation, Eq. (18), the corresponding kernels read

$$\Gamma_X(t) = K(t)\left\{\left(\frac{\Delta}{\varphi}\right)^2+\left(\frac{\omega_A}{\varphi}\right)^2\cos[\varphi t]\right\}, \quad \text{(A5a)}$$

$$\Gamma_Y(t) = K(t)\cos[\varphi t], \quad \text{(A5b)}$$

$$\Upsilon(t) = K(t)\frac{\omega_A}{\varphi}\sin[\varphi t], \quad \text{(A5c)}$$

$$\Phi_X(t) = K(t)\frac{\omega_A\Delta}{\varphi^2}\{1-\cos[\varphi t]\}, \quad \text{(A5d)}$$

$$\Phi_Y(t) = K(t)\frac{\Delta}{\varphi}\sin[\varphi t], \quad \text{(A5e)}$$

where $\varphi=\sqrt{\omega_A^2+\Delta^2}$. The extra kernels $\Phi_X(t)$ and $\Phi_Y(t)$ couple the derivative of $S_Z(t)$ to the averages $S_X(t)$ and $S_Y(t)$, respectively, i.e., $dS_Z(t)/dt=\Delta S_Y(t)-\int_0^t d\tau\{\Phi_X(t-\tau)S_X(\tau)+\Phi_Y(t-\tau)S_Y(\tau)\}$. For the exact evolution, these kernels vanish.

APPENDIX B: KERNELS IN THE SPIN-BOSON-MODEL NOTATION

The kernels of the spin-boson model Eqs. (43a)–(43c), in lowest order in Δ, read [22]

$$Y_A^{(s)}(t) \simeq -\Delta Y_B^{(s)}(t)\sin[Q''(t)], \quad \text{(B1a)}$$

$$K_A^{(a)}(t) \simeq \Delta^2 Y_B^{(a)}(t)\sin[Q''(t)], \quad \text{(B1b)}$$

$$K_A^{(s)}(t) \simeq \Delta^2 Y_B^{(s)}(t)\cos[Q''(t)], \quad \text{(B1c)}$$

$$Y_A^{(a)}(t) \simeq -\Delta Y_B^{(a)}(t)\cos[Q''(t)], \quad \text{(B1d)}$$

$$K_B^{(s)}(t) \simeq \Delta Y_B^{(s)}(t), \quad \text{(B1e)}$$

$$K_B^{(a)}(t) \simeq -\Delta Y_B^{(a)}(t), \quad \text{(B1f)}$$

$$Y_B^{(s)}(t) \simeq \cos[\omega_A t]e^{-Q'(t)}, \quad \text{(B1g)}$$

$$Y_B^{(a)}(t) \simeq -\sin[\omega_A t]e^{-Q'(t)}, \quad \text{(B1h)}$$

where $Q'(t)$ and $Q''(t)$ are defined by

$$Q'(t) = \frac{d^2}{\hbar\pi}\int_0^\infty dw\frac{J(w)}{w^2}\coth\left(\frac{\hbar w}{2kT}\right)[1-\cos(wt)], \quad \text{(B2)}$$

$$Q''(t) = \frac{d^2}{\hbar\pi}\int_0^\infty dw\frac{J(w)}{w^2}\sin(wt). \quad \text{(B3)}$$

The exact evolution Eq. (24a)–(24c) can be written as in Eq. (43a)–(43c) with the definitions

$$Y_A^{(s)}(u) = K_A^{(a)}(u) = 0, \quad \text{(B4a)}$$

$$Y_A^{(s)}(u) = K_A^{(a)}(u) = 0, \quad \text{(B4b)}$$

$$K_A^{(s)}(u) = T(u)\Delta^2[u+\Gamma_X(u)], \quad \text{(B4c)}$$

$$Y_A^{(a)}(u) = T(u)\Delta[\omega_A-\Upsilon(u)], \quad \text{(B4d)}$$

$$Y_B^{(s)}(u) = T(u)\Delta[u + \Gamma_X(u)], \tag{B4e}$$

$$K_B^{(a)}(u) = T(u)\Delta[\omega_A - \Upsilon(u)], \tag{B4f}$$

$$Y_B^{(s)}(u) = T(u)[u + \Gamma_Y(u)], \tag{B4g}$$

$$Y_B^{(a)}(u) = -T(u)[\omega_A - \Upsilon(u)], \tag{B4h}$$

where we have introduced

$$T(u) = \frac{1}{[\omega_A - \Upsilon(u)]^2 + [u + \Gamma_X(u)][u + \Gamma_Y(u)]}. \tag{B5}$$

The structure of these kernels is the same as those of the spin-boson model in the limit of vanishing Δ, which implies that Δ appears only through the unitary evolution. In fact, in this limit we can approximate Eqs. (A1a)–(A1c) by $\Gamma_X(u) \simeq \Gamma_Y(u) \simeq [K(u-i\omega_A)+K(u+i\omega_A)]/2$ and $\Upsilon(u) \simeq [K(u-i\omega_A) - K(u+i\omega_A)]/2i$. After introducing these expressions in Eqs. (B4a)–(B4h), it is simple to get $Y_B^{(s)}(t) \simeq \cos[\omega_A t]P_0(t)$, and $Y_B^{(a)}(t) \simeq -\sin[\omega_A t]P_0(t)$. Then, disregarding in Eqs. (B1a)–(B1h) the phase contribution proportional to $Q''(t)$, which is valid in the high temperature limit [29], a mapping with Eqs. (B4a)–(B4h) can be done after imposing the equality $P_0(t) = e^{-Q'(t)}$.

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[39] Notice that when a bath can be split as an external product of Markovian subreservoirs, the corresponding reduced system dynamics is Markovian.

[40] The stochastic dynamics can be formally extended to the case $\mathcal{L} \neq \mathcal{E} - I$ [30] after introducing the superoperator $\mathcal{E}[\rho] = \{I + [e^{\kappa \mathcal{L}} - I]\}\rho$, where κ must be intended as a control parameter. Then, the evolution Eq. (18) is regained in the limit in which simultaneously $\kappa \to 0$ and the number of events per unit of time go to infinity, the last limit being controlled by the sprinkling distribution $f(t)$ [38]. This object is defined as the probability density for an event at time t, disregarding the possibility of extra events in $(0,t)$. Therefore, it is given by $f(t) = w(t)\theta(t) + \int_0^t w(t-\tau) f(\tau)$, where $\theta(t)$ is the step function [here defined as $\theta(t) = 1$ for $t > 0$, and $\theta(t) = 0$ for $t \leqslant 0$, which implies $f(t)|_{t=0} \equiv 0$]. In the Laplace domain it reads $f(u) = w(u)/[1 - w(u)]$, which allows us to write $K(t) = df(t)/dt$.

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[44] When the random rate assumes only two different values $\gamma_{\uparrow/\downarrow}$, with probabilities $P_{\uparrow/\downarrow}$, the exact kernel reads $K(u) = \langle \gamma \rangle / [1 + \beta \sigma(u)]$ where $\sigma(u) = 1/[u + \eta/(\langle \gamma \rangle \langle \tau \rangle)]$. The rate β is defined in terms of the fluctuations $\beta = [\langle \gamma_R^2 \rangle - \langle \gamma_R \rangle^2]/\langle \gamma_R \rangle$. The extra rates are $\eta = P_\uparrow \gamma_\downarrow + P_\downarrow \gamma_\uparrow$, and $\langle \tau \rangle = P_\uparrow / \gamma_\uparrow + P_\downarrow / \gamma_\downarrow$.

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